Scattering operators on Fock space. I. Compact groups and internal symmetries

1985 J. Phys. A: Math. Gen. 18 3141

(http://iopscience.iop.org/0305-4470/18/16/016)

J. Phys. A: Math. Gen. 18 (1985) 3141-3150. Printed in Great Britain

Scattering operators on Fock space: I. Compact groups and internal symmetries

W H Klink†

Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA

Received 7 March 1985

Abstract. The symmetric Fock space for a given representation of a compact internal symmetry group is shown to be naturally associated with an induced representation space, induced by a unitary group. The induced representation space is a Hilbert space over the complex sphere, and the orbits on this complex sphere with respect to the internal symmetry group give rise to a double coset decomposition. A scattering operator acting on double coset representatives only is shown to be unitary, invariant with respect to the internal symmetry group, and include particle production. A simple example in which positively and negatively charged mesons generate a Fock space with respect to a U(1) internal symmetry group is given.

1. Introduction

Because the strong nuclear forces are short range, a natural theoretical quantity with which to investigate these forces is the scattering operator. Whether it is viewed as a derived quantity, as in the case of quantum field theory, or is viewed as fundamental, as in analytic S matrix theory, the scattering operator provides a direct link between experimental results and theoretical considerations. Of particular interest is that any relativistic scattering operator must necessarily deal with production and other multiparticle phenomena.

Many properties of the scattering operator are model independent, in the sense that however the scattering operator is obtained, these properties must still hold. Such properties include, but are not exhausted by, unitarity, which expresses the conservation of probability, invariance with respect to an underlying group, from which the conservation laws arise, crossing, which deals with the interplay of particles and antiparticles, relativistic causality and cluster properties.

Further, many of the model-independent properties of the scattering operator have a group theoretical origin—the invariance of the scattering operator is an obvious example. The group in question is generally of the form P × K, where P is the Poincaré group, reflecting the spacetime properties of the scattering operator, while K is a compact group such as SU(2) or SU(3), reflecting the internal symmetry properties of the scattering operator. The goal of this series of papers is to find a representation of the scattering operator for which the above mentioned properties are automatically satisfied. Some of these properties—such as relativistic causality and cluster properties—are associated only with the Poincaré group. Other properties—such as unitarity,

† This work was supported in part by the Department of Energy.

invariance and crossing—are associated with both the Poincaré group and the internal symmetry group. For example, the invariance of the scattering operator under $P \times K$ means that its kernel is diagonal in energy and momentum—reflecting the representation structure of the Poincaré group—and diagonal in the irreducible representations of the internal symmetry group.

For all types of strongly interacting particles—such as pions and nucleons—the representation of the scattering operator should automatically include production and other many-body phenomena, as well as the bosonic or fermionic nature of the particles. It is therefore natural to choose as a starting point the many-particle Fock space generated by the underlying representation of the group. But such a space is generally not well suited for building in the desired properties of the scattering operator. So it will be necessary to find other spaces which, while not as physically transparent as the Fock space nevertheless, make it easier to deal with the physical requirements imposed on the scattering operator.

In this first paper attention will be focused exclusively on internal symmetries and all spacetime dependence of the scattering operator will be suppressed. The Fock space will then be the many-particle space generated by a multiplet of bosons such as the three-pion multiplet for isospin symmetry or the eight-meson multiplet associated with SU(3) flavour. We will exhibit a representation of the scattering operator on this many-particle internal symmetry Fock space which is unitary, invariant with respect to the internal symmetry, and has particle production. Crossing properties with respect to internal symmetries will be discussed in a future paper.

The main reason for not dealing with the Poincaré group is that the mathematical analysis is more complicated for a non-compact group such as the Poincaré group, or even $P \times K$, than with a compact group alone. We hope, however, in succeeding papers to generalise the techniques introduced here and thereby find representations for scattering operators that are relativistically invariant, unitary, and have particle production.

Moreover, to have representations of the scattering operator with respect to a compact internal symmetry is of interest because there are physical consequences of internal symmetries independent of spacetime considerations. For example, if the internal symmetry is charge, with U(1) the associated internal symmetry group, and the Fock space is generated by the two-dimensional space consisting of $+1$ and -1 charges, then the Fock space can be interpreted as the many particle space of positively and negatively charged pions. A representation of the scattering operator for this example is given in § 4.

For the case of compact internal symmetries, there are two key steps needed to construct representations of the scattering operator. First, for bosons the Fock space generated by the internal symmetry is shown to be isomorphic to the Fock space generated by the fundamental representation of a unitary group; we show that this Fock space in turn can naturally be associated with an induced representation space of the unitary group. The analysis of these spaces is carried out in § 2.

The induced representation space is actually a Hilbert space over the complex sphere. The internal symmetry scattering operator is then defined to be a unitary operator on this Hilbert space over the complex sphere. If the scattering operator acts only on the orbits of the complex sphere (with respect to the internal symmetry group), we show that the scattering operator is invariant with respect to the underlying internal symmetry group. Stated differently, if the scattering operator acts only on double coset representatives, then it will be invariant with respect to the underlying internal symmetry group. When transformed back to the original Fock space, the scattering operator is shown to have all the desirable physical properties mentioned earlier. This part of the analysis is carried out in § 3, while the connection with the complex sphere is given in the appendix.

2. $S(V)$ is equivalent to an induced representation

Let V be the (not necessarily irreducible) representation space of a compact group K, of dimension N. Then the symmetric Fock space is

$$S(V) = \sum_{n=0}^{\infty} \oplus V_n, \tag{1}$$

where V_n is the n-fold symmetric tensor product, $V_n = (V\otimes \ldots \otimes V)_{\text{sym}}$.

Now consider the fundamental representation space $V^{(10\ldots0)}$ of $\mathrm{U}(N)$ (Klink and Ton-That 1982, Barut and Raczka 1977†); it is also of dimension N, so that

$$S(V) = S(V^{(10\ldots0)}).$$

However, it is known (Weyl 1946) that the n-fold symmetric tensor product of $V^{(10\ldots0)}$ is simply $V^{(n0\ldots0)}$; that is

$$V_n^{(10\ldots0)} = V^{(n0\ldots0)}, \tag{2}$$

where $(n0\ldots0)$ is an irreducible representation of $\mathrm{U}(N)$. Thus

$$S(V) = \sum_{n=0}^{\infty} \oplus V^{(n0\ldots0)}. \tag{3}$$

But $\Sigma_{n=0}^{\infty}\oplus V^{(n0\ldots0)}$ can be written as an induced representation space (Barut and Raczka 1977). To see this, consider the induced representation of $\mathrm{U}(N)$ induced from the identity representation of $\mathrm{U}(N-1)$. The representation space on which this induced representation acts is denoted by $L^2[\mathrm{U}(N)]/[\mathrm{U}(N-1)\to 1]$ and given by

$$L^2\left(\frac{\mathrm{U}(N)}{\mathrm{U}(N-1)\to 1}\right) \equiv \{f\colon \mathrm{U}(N)\to C|f(hg)=f(g),$$
$$\|f\|^2 = \int \mathrm{d}g|f(g)|^2 < \infty,\ g\in \mathrm{U}(N),\ h\in \mathrm{U}(N-1)\}. \tag{4}$$

The number of times an irreducible representation $(m) = (m_1 \ldots m_N)$ of $\mathrm{U}(N)$ occurs in $L^2[\mathrm{U}(N)]/[\mathrm{U}(N-1)\to 1]$ can be computed using the Frobenius reciprocity theorem (Barut and Raczka 1977):

multiplicity ((m) in L^2)
= multiplicity (identity representation of $\mathrm{U}(N-1)$ in
(m) of $\mathrm{U}(N)$ restricted to $\mathrm{U}(N-1)$).

But from the Gelfand betweenness relations (Klink and Ton-That 1982, Barut and Raczka 1977), the number of times the identity representation of $\mathrm{U}(N-1)$, $(0\ldots0)$ occurs in $(m) = (m_1, \ldots, m_N)$ is 0 unless $(m) = (n, 0, \ldots, 0)$, and then the multiplicity

† For a more general discussion of the $\mathrm{U}(N)$ groups see especially chapters 8 and 10.

is 1. It follows that

$$S(V^{(10\dots0)}) \simeq L^2\left(\frac{U(N)}{U(N-1)\to 1}\right). \tag{5}$$

Thus, it is seen that the symmetric Fock space of V (or $V^{(10\dots0)}$), $S(V)$, is isomorphic in a natural way to an induced representation space $L^2[U(N)]/[U(N-1)\to 1]$.

Though the spaces V (the representation space of K) and $V^{(10\dots0)}$ (the fundamental representation space of $U(N)$) are isomorphic because they have the same (finite) dimension, the bases may look very different for the two spaces. Since K is a compact group of internal symmetries, there is a basis dictated by physical considerations for V, which is denoted by e_i, $i=1\dots N$. With this basis one can compute the matrix elements

$$D_{ii'}(k) = (e_i, U_k e_{i'}), \qquad k \in K, \qquad U_k \text{ the representation of K on } V, \tag{6}$$

which are $N\times N$ unitary matrices. Thus, $D_{ii'}(k)\in U(N)$ and $D_{ii'}(K)$ is a subgroup of $U(N)$.

Since $D_{ii'}(K)$ is a subgroup of $U(N)$, the space $V^{(n0\dots0)}$ can be reduced to irreducible representations of $D_{ii'}(K)$, so that

$$V^{(n0\dots0)} = \sum_{\chi,\eta} \oplus V^{\chi}_{\eta,n}, \tag{7}$$

where χ is an irreducible representation of $D_{ii'}(K)$ and η a multiplicity label. Finally,

$$S(V) = \sum_{n=0}^{\infty} \oplus V^{(n0\dots0)} = \sum_{n,\chi\eta} \oplus V^{\chi}_{\eta,n} \tag{8}$$

gives the decomposition of the symmetric Fock space into irreducible representations of K via representations of $U(N)$.

Actually, such a decomposition of $S(V)$ is independent of $U(N)$ and depends only on how symmetric tensor products of V decompose into irreducible representations of K.

3. The scattering operator on $S(V)$

In analogy with the usual scattering operator, which maps the Fock space of a representation of the Poincaré group into itself, we define S as a unitary operator from $S(V)$ to $S(V)$ satisfying 'relativistic invariance':

$$SU_k = U_k S \qquad \text{on } S(V), \tag{9}$$

where $k\in K$ and U_k is the representation of K on $S(V)$ inherited from the representation of K on V. What is here called 'relativistic' invariance is seen to be simply the statement that S commutes with U_k.

The goal of this section is to find a natural representation of S on $S(V)$ so that S is automatically unitary, commutes with K, and contains 'particle production'; that is, we want a representation of S not diagonal in n—there should be a mixing of particle number.

To find such a representation of S, it is most natural to work with the space $L^2[U(N)]/[U(N-1)\to 1]$. Now L^2 was defined in § 2 as an induced representation space; the induced (reducible) representation of U(N) is given by right translation:

$$(R_{g_0}f)(g)=f(gg_0), \qquad g_0\in \mathrm{U}(N) \qquad f\in L^2\left(\frac{\mathrm{U}(N)}{\mathrm{U}(N-1)\to 1}\right). \tag{10}$$

Consider next the double coset decomposition of U(N) given by

$$\mathrm{U}(N-1)\backslash(\mathrm{U}(N)/\mathrm{D}_{ii'}(K))\equiv \mathrm{U}(N-1)\backslash(\mathrm{U}(N)/\mathrm{D}(K)).$$

Define the map T_{D} from L^2 to representation spaces of K by

$$(T_{\mathrm{D}}f)(k)=f(g_{\mathrm{D}}\mathrm{D}(k)), \tag{11}$$

where g_{D} is a double coset representative. Then $T_{\mathrm{D}}L^2$ forms a representation space of K, with the representation given by

$$(\tilde{R}_{k_0}T_{\mathrm{D}}f)(k)\equiv(T_{\mathrm{D}}R_{k_0}f)(k)=(R_{k_0}f)(g_{\mathrm{D}}\mathrm{D}(k))=f(g_{\mathrm{D}}\mathrm{D}(kk_0))=(T_{\mathrm{D}}f)(kk_0), \tag{12}$$

k_0 an element of K, so that

$$L^2\left(\frac{\mathrm{U}(N)}{\mathrm{U}(N-1)\to 1}\right)\cong\int_{\oplus}\mathrm{d}\mu(D)(T_{\mathrm{D}}L^2),$$

where $\mathrm{d}\mu(D)$ is the measure associated with the double cosets.

Now define S on $T_{\mathrm{D}}f\equiv f_{\mathrm{D}}$ as

$$(Sf_{\mathrm{D}})(k)\equiv\int \mathrm{d}\mu(D')K(D,D')f_{\mathrm{D}'}(k), \tag{13}$$

where $K(D, D')$ is a kernel depending only on double coset labels. It is easy to see that S commutes with $\tilde{R}_{k_0}$:

$$\begin{aligned}(\tilde{R}_{k_0}Sf_{\mathrm{D}})(k)&=(Sf_{\mathrm{D}})(kk_0)\\&=\int \mathrm{d}\mu(D')K(D,D')f_{\mathrm{D}'}(kk_0)\\(S\tilde{R}_{k_0}f_{\mathrm{D}})&=\int \mathrm{d}\mu(D')K(D,D')(\tilde{R}_{k_0}f_{\mathrm{D}'})(k)\\&=\int \mathrm{d}\mu(D')K(D,D')f_{\mathrm{D}'}(kk_0).\end{aligned} \tag{14}$$

Further, the kernel $K(D, D')$ can easily be made unitary with respect to $\mathrm{d}\mu(D)$. Thus, the representation of S given in equation (13) can be made unitary and commutes with the action of K.

To show how S changes the number of particles, it is most useful to go to the 'partial wave' space $V^{\chi}_{\eta,n}$, given in equation (7). We want to show that, in general, the action of S on $V^{\chi}_{\eta,n}$ changes the particle number n.

To that end, we want to find a map from $T_{\mathrm{D}}L^2$ to $V^{\chi}_{\eta,n}$. Define the mixed basis matrix element of U(N) as

$$D^{(n)}_{[0]\chi\eta i}(g)=\langle(n0\ldots0),[0]|R_g|(n0\ldots0)\chi\eta i\rangle, \tag{15}$$

where (n) stands for the irreducible representation $(n0\ldots0)$ of U(N), [0] stands for the Gelfand pattern $\begin{smallmatrix}0 & \cdots & 0\\ & \ddots & \\ & 0 & \end{smallmatrix}$ and $\chi\eta i$ are the labels of the irreducible representation, multiplicity and basis, respectively, of the subgroup D(K) of U(N). How one constructs such a matrix element will be discussed in the appendix.

Notice that $D^{(n)}_{[0]\chi\eta i}(g)$ is an element of $L^2[\mathrm{U}(N)]/[\mathrm{U}(N-1)\to 1]$, since it transforms to the left as the identity representation of $\mathrm{U}(N-1)$ and is square integrable with respect to the Haar measure of U(N). Therefore, it makes sense to write $(T_{\mathrm{D}}D^{(n)}_{[0]\chi\eta i})(k)$. The mapping Λ from L^2 to $S(V)$ is then given by

$$\begin{aligned}(\Lambda f)(n,\chi\eta i)&=\int \mathrm{d}g\, D^{(n)*}_{[0]\chi\eta i}(g)f(g)\\&=\int \mathrm{d}\mu(D)\,\mathrm{d}(k)D^{(n)*}_{[0]\chi\eta i}(g_{\mathrm{D}}k)f(g_{\mathrm{D}}k)\\&\equiv F(n,\chi\eta i),\qquad f\in L^2,\end{aligned}\tag{16}$$

and the action of S on elements F of $V^{\chi}_{\eta,n}$ is given by

$$\begin{aligned}(SF)(n,\chi\eta i)&=(S\Lambda f)(n,\chi\eta i)\\&\equiv(\Lambda Sf)(n,\chi\eta i)\\&=\int \mathrm{d}\mu(D)\,\mathrm{d}(k)D^{(n)*}_{[0]\chi\eta i}(g_{\mathrm{D}}k)\int \mathrm{d}\mu(D')K(D,D')f(g_{\mathrm{D}'}k)\\&=\int \mathrm{d}\mu(D)\,\mathrm{d}\mu(D')\,\mathrm{d}(k)D^{(n)*}_{[0]\chi\eta i}(g_{\mathrm{D}}k)\\&\quad\times K(D,D')\sum_{n'\chi'\eta' i'}D^{(n')}_{[0]\chi'\eta' i'}(g_{\mathrm{D}'}k)F(n',\chi',\eta' i')\\&=\sum_{\substack{n'\chi'\eta' i'\\ \bar{i}\bar{i}'}}F(n',\chi'\eta' i')\int \mathrm{d}k\,\mathrm{d}\mu(D)\,\mathrm{d}\mu(D')D^{(n)*}_{[0]\chi\eta\bar{i}}(g_{\mathrm{D}})\\&\quad\times D^{\chi^*}_{\bar{i}i}(k)D^{(n)}_{[0]\chi'\eta'\bar{i}'}(g_{\mathrm{D}'})D^{\chi'}_{\bar{i}'i'}(k)K(D,D')\\&=\sum_{n'\eta'\bar{i}}F(n'\chi\eta' i)\int \mathrm{d}\mu(D)\,\mathrm{d}\mu(D')D^{(n)*}_{[0]\chi\eta\bar{i}}(g_{\mathrm{D}})K(D,D')D^{(n')}_{[0]\chi\eta'\bar{i}}(g_{\mathrm{D}'})\\&=\sum_{n'\eta'}K_{\chi}(n\eta,n'\eta')F(n',\chi\eta' i),\end{aligned}\tag{17}$$

where the kernel of S is given by

$$K_{\chi}(n\eta,n'\eta')=\sum_{\bar{i}}\int \mathrm{d}\mu(D)\,\mathrm{d}\mu(D')D^{(n)*}_{[0]\chi\eta\bar{i}}(g_{\mathrm{D}})K(D,D')D^{(n')}_{[0]\chi\eta'\bar{i}}(g_{\mathrm{D}'}).\tag{18}$$

It is seen that the kernel $K_{\chi}(n\eta,n'\eta')$ is diagonal in χ ('relativistic' invariance) and pushes n, the number of particles, and η, the degeneracy parameters (which would correspond to subenergies if K could be replaced by the Poincaré group) around to different values $n'\eta'$. Further $K_{\chi}(n\eta,n'\eta')$ can be made unitary with respect to the labels $n\eta$, $n'\eta'$. So we have constructed a representation of the scattering operator which is unitary, 'relativistically' invariant, and contains 'particle production', at least with respect to any internal symmetries generated by the representation of a compact group.

4. A simple example

We consider in this section a U(1) internal symmetry corresponding to charge and acting on a two-dimensional representation space V consisting of +1 and −1 charges. The two elements in V can be thought of as a positive and negative pi meson; then $S(V)$ is the space of all possible combinations of π^+ and π^- particles.

Since V is a two-dimensional (reducible) representation space of U(1), the unitary group with a fundamental representation that is two dimensional is U(2), and its fundamental representation space is written $V^{(10)}$.

The matrix element D(k) of equation (6) is given by

$$\mathrm{D}_{ii'}(\theta)=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\theta} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\theta}\end{pmatrix}\equiv \mathrm{D}(\theta)\in \mathrm{U}(2) \tag{19}$$

and in order to compute the mixed basis matrix element of equation (15), it is necessary to find the charge content Q of the irreducible representation (n0) of U(2). Now a polynomial basis for the (n0) representation of U(2) is given by Klink and Ton-That (1982) and Barut and Raczka (1977)

$$e_k^{(n0)}(g)=\frac{g_{11}^k g_{12}^{n-k}}{[k!(n-k)!]^{1/2}},\qquad g\in \mathrm{GL}(2,\boldsymbol{C}),\qquad k=0,\dots n, \tag{20}$$

with

$$(R_{g_0}e_k^{(n0)})(g)=e_k^{(n0)}(gg_0),\qquad g_0\in \mathrm{GL}(2,\boldsymbol{C}).$$

In particular,

$$R_{\mathrm{D}(\theta)}e_k^{(n0)}(g)=\exp[\mathrm{i}(2k-n)\theta]e_k^{(n0)}(g) \tag{21}$$

which means that the charge content of the (n0) representation is $Q=2k-n$, $k=0,\dots,n$, or $Q=-n,-n+2,\dots,n-2,n$. A charge state in $V^{(n0)}$ can thus be written as

$$|(n0),Q\rangle\equiv e_Q^{(n0)}(g)=g_{11}^{(n+Q)/2}g_{12}^{(n-Q)/2}\left[\left(\frac{n+Q}{2}\right)!\left(\frac{n-Q}{2}\right)!\right]^{-1/2} \tag{22}$$

and for a given n, Q, the number of positive pions n_+ and negative pions n_- is given by $n_+ + n_- = n$, $n_+ - n_- = Q$, so that (22) can also be written as

$$e_Q^{(n0)}=g_{11}^{n_+}g_{12}^{n_-}(n_+!\,n_-!)^{-1/2}.$$

We now turn to the computation of the double cosets. The relevant homogeneous space is [U(2)]/[U(1)], consisting of elements $\binom{z_1}{z_2}=z$, with $z^\dagger z=1$ and a measure given by $\mathrm{d}\mu(z)=\mathrm{d}z_1\,\mathrm{d}z_2\,\delta(z^\dagger z-1)$. The orbits of Z_2 with respect to $D(\mathrm{U}(1))$ will then give the double coset representatives needed for the mixed basis matrix element. Since $\mathrm{D}(\theta)=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\theta} & \\ & \mathrm{e}^{-\mathrm{i}\theta}\end{pmatrix}$, it is clear that U(1) acts only on the phases of $z=\binom{z_1}{z_2}$; writing

$$z=\begin{pmatrix}\sin\alpha & \mathrm{e}^{\mathrm{i}\phi_1}\\ \cos\alpha & \mathrm{e}^{\mathrm{i}\phi_2}\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{\mathrm{i}\theta} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}\theta}\end{pmatrix}\begin{pmatrix}\sin\alpha & \mathrm{e}^{\mathrm{i}\phi}\\ \cos\alpha & \mathrm{e}^{\mathrm{i}\phi}\end{pmatrix},\qquad 0\leqslant\alpha\leqslant\tfrac{1}{2}\pi \tag{23}$$

gives $\phi_1=\phi+\theta$, $\phi_2=\phi-\theta$, with

$$\int_0^{2\pi}\mathrm{d}\phi_1\,\mathrm{d}\phi_2=\int_{-\pi}^{+\pi}\mathrm{d}\theta\int_{|\theta|}^{2\pi-|\theta|}2\,\mathrm{d}\phi,$$

so the double coset measure is

$$\int \mathrm{d}\mu(D) = \int_0^{\pi/2} \sin 2\alpha \, \mathrm{d}\alpha \int_0^{2\pi} \mathrm{d}\phi \tag{24}$$

and the double coset representative can be chosen as

$$g_{\mathrm{D}} = \mathrm{e}^{\mathrm{i}\phi}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{U}(2). \tag{25}$$

Finally, the mixed basis matrix element

$$D_{0Q}^{(n0)}(g_{\mathrm{D}}) \equiv \langle (n0), [0] | R_{g_{\mathrm{D}}} | (n0) Q \rangle \tag{26}$$

is needed to compute the kernel, equation (18). This is a standard SU(2) matrix element and will not be given here. The kernel becomes

$$K_Q(n, n') = \int \mathrm{d}\mu(D)\, \mathrm{d}\mu(D') D_{0Q}^{(n0)*}(g_{\mathrm{D}}) K(D, D') D_{0Q}^{(n'0)}(g'_{\mathrm{D}}) \tag{27}$$

with $K(D, D') = K(\alpha, \phi; \alpha'\phi')$ any unitary 'matrix' with respect to the measure (24).

5. Conclusion

We have exhibited a representation of the scattering operator for compact internal symmetry groups that is invariant with respect to an internal symmetry group K, unitary, mixes particle number and preserves the bosonic character of the underlying particles. The representation is given in terms of a kernel over double coset parameters; the double coset representatives themselves come from $\mathrm{U}(N-1)\backslash\mathrm{U}(N)/\mathrm{D}(K)$, or what is equivalent, the orbits of the complex N sphere with respect to the subgroup $\mathrm{D}(K)$ of $\mathrm{U}(N)$. Except for the unitarity requirement, this kernel over double cosets is quite arbitrary, so other physical requirements are needed to further constrain the representation.

One such requirement is crossing symmetry, whereby the amplitudes for direct and crossed channels of multiparticle reactions are related. In succeeding papers further physical requirements such as crossing symmetry will be analysed to see how they constrain the set of unitary invariant scattering operators.

Though the formalism presented here is general in the sense that any internal symmetry group K with representation space V generates a representation of the scattering operator, what is of most physical interest is when $\mathrm{K} = \mathrm{SU}(3)_{\text{flavour}}$ and V is the eight-dimensional representation of SU(3). What must then be computed are the matrix elements of SU(3) in the eight-dimensional representation, so that the orbits of the eight-dimensional complex sphere with respect to these matrix elements can be obtained. The resulting double coset parameters can then be used in the mixed basis matrix elements $D_{[0]\chi\eta i}^{(n0\ldots0)}(g_{\mathrm{D}})$, where χ is an irreducible representation of SU(3) and η a multiplicity label arising from the reduction of the $(n0\ldots0)$ representation of U(8) to SU(3) representations.

In this paper only symmetric Fock spaces associated with bosons have been considered. Of obvious interest are fermion Fock spaces arising from compact internal symmetries such as SU(2) or SU(3). Since such fermion Fock spaces are finite dimensional—in contrast to the boson Fock spaces considered in this paper—techniques other than those given in this paper will be needed to construct unitary invariant scattering operators. These topics will be discussed in a succeeding paper.

Finally, of course, one would like to generalise the techniques introduced in this paper to non-compact groups such as the Poincaré group. It should be pointed out that all known solutions of model field theories give scattering operators that do not allow for particle production (Ruisenaars 1980). So it is of obvious interest to find representations of the scattering operator on a Fock space generated by representations of the Poincaré group P, or P×K, that do allow for particle production, as was the case for the scattering operators, equation (18), for internal symmetries. This topic will also be discussed in a succeeding paper.

Appendix. The homogeneous space U(N)/U($N-1$), double cosets and mixed basis matrix elements

To find the double cosets $U(N-1)\backslash U(N)/D(K)$, it is most convenient to introduce the homogeneous space $U(N)/U(N-1)$ and then see how elements $D(k)$, $k \in K$ push around points in $U(N)/U(N-1)$. Now $U(N)/U(N-1)$ can be realised as the complex sphere; that is, let z be the column vector $z_1 \ldots z_N$, and consider the complex manifold

$$Z_N \equiv \left\{ z^\dagger z = 1,\ z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix} \right\} \tag{A1}$$

with the measure $d\mu(z) = dz_1 \ldots dz_N\, \delta(z^\dagger z - 1)$. Then $gz \in Z_N$ for $g \in U(N)$ and if the stability point is chosen as $z_0 = \binom{0}{1}$, it is clear that the subgroup $U(N-1)$, imbedded in $U(N)$ by $\begin{pmatrix} U(N-1) & 0 \\ 0 & 1 \end{pmatrix}$ leaves z_0 invariant.

Now $D(K)$ is the subgroup of $U(N)$ obtained from unitary matrix elements of K. Since $D(k) \in U(N)$, $k \in K$, $D(k)z$ is again an element in Z_N; in particular the subgroup of K leaving an arbitrary point $z \in Z_N$ invariant, i.e., $D(k)z = z$, is useful in actually choosing double coset representatives, for the elements of this subgroup can be ignored, while the remaining (coset) elements of K do in fact push points z around.

The connection between the action of the group $U(N)$ on Z_N and the induced representation space introduced in (4) is given in the following way: the representation space of $U(N)$ induced by the identity representation of $U(N-1)$ can be written as a representation space over $[U(N)]/[U(N-1)] = Z_N$. Let $f \in L^2\{[U(N)]/[U(N-1) \to 1]\}$ and define a map F from L^2 to functions over Z_N by

$$(Ff)(z) = f(g^{-1}(z)), \qquad f \in L^2\left(\frac{U(N)}{U(N-1) \to 1}\right), \tag{A2}$$

where $z \in Z_N$, and $g(z)$ is a coset representative satisfying $z = g(z)z_0$, $g \in U(N)$. The induced representation R_{g_0} on L^2 becomes

$$\begin{aligned}(\hat{R}_{g_0}Ff)(z) &\equiv (FR_{g_0}f)(z) = (R_{g_0}f)(g^{-1}(z)) = f(g^{-1}(z)g_0) \\ &= f(hg^{-1}(z')) = f(g^{-1}(z')) = (Ff)(z') = (Ff)(g_0^{-1}z),\end{aligned} \tag{A3}$$

where $g_0 \in U(N)$, $h \in U(N-1)$ and use has been made of the fact that $f(hg) = f(g)$,

$f \in L^2$. Further

$$g^{-1}(z)g_0 = hg^{-1}(z') \qquad g_0 g(z') = g(z)h \tag{A4}$$

and applying z_0 to both sides of (A4) gives

$$g_0 g(z')z_0 = g(z)hz_0 \qquad g_0 z' = g(z)z_0 = z \tag{A5}$$

so $z' = g_0^{-1}z$. Thus with our convention, with the induced representation being given by right translation (see equation (10)), the action of the group on manifold points $z \in Z_N$ is given by the inverse element, equation (A3).

This finally leads to the computation of mixed basis matrix elements needed for the kernel of the S operator. Since $\mathrm{D}(K)$ is a subgroup of $\mathrm{U}(N)$, one can ask how the irreducible representation $(n0\ldots0)$ of $\mathrm{U}(N)$ decomposes into irreducible representations of K. This decomposition was written as

$$V^{(n0\ldots0)} = \sum_{\chi,\eta} \oplus V^{\chi}_{\eta,n} \tag{7}$$

in § 2, where χ is an irreducible representation of K and η a multiplicity label. Then it is possible to choose a basis in $V^{(n0\ldots0)}$ that involves χ, η and a basis set i depending only on the irreducible representations of K. Write this basis $|(n0\ldots0);\ \chi\eta i\rangle$. We wish to compute the matrix element

$$D^{(n)}_{[0]\chi\eta i}(\gamma) \equiv \langle (n0\ldots0);[0]|R_\gamma|(n0\ldots0);\chi\eta i\rangle, \tag{A6}$$

where $\gamma \in \mathrm{GL}(n, \boldsymbol{C})$. (It is easiest to compute matrix elements for $\mathrm{GL}(n, \boldsymbol{C})$ and then restrict to elements of $\mathrm{U}(N)$.)

Now the basis elements $(n0\ldots0);\ [0]\rangle$ are most easily realised as functions over $\mathrm{GL}(n, \boldsymbol{C})$; in particular

$$h^{(n0\ldots0)}_{[0]}(\gamma) = \gamma^n_{1N}, \qquad \gamma \in \mathrm{GL}(N, \boldsymbol{C}).$$

Using the differentiation inner product gives the normalised polynomial basis element

$$e^{(n0\ldots0)}_{[0]}(\gamma) = (n!)^{-1/2}\gamma^n_{1N}. \tag{A7}$$

The harder basis element to compute is $|(n0\ldots0);\ \chi\eta i\rangle$. Without specifying the compact group K and the representation space V, it is difficult to concretely realise such a basis element.

References

Barut A O and Raczka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN-Polish Scientific Publishers)

Klink W H and Ton-That T 1982 *Ann. Inst. Henri Poincaré* **36A** 225

Ruisenaars S N M 1980 *Ann. Phys., NY* **128** 335

Weyl H 1946 *The Classical Group, Their Invariants and Representations* (Princeton, NJ: Princeton University Press)